

# Discrete time approximation of coalescing stochastic flows on the real line

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**Abstract.** In this paper we have constructed an approximation for the Harris flow and the Arratia flow using a sequence of independent stationary Gaussian processes as a perturbation. We have established what should be the relationship between the step of approximation and smoothness of the covariance of the perturbing processes in order to have convergence of the approximating functions to the Arratia flow.

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**Key words:** Stochastic flow, stochastic differential equation, numerical approximation.

**Introduction.** It is well-known [1], that the solution to the Cauchy problem for SDE

$$\begin{cases} dx(t) = a(x(t))dt + b(x(t))dw(t) \\ x(0) = u_0 \end{cases} \quad (1)$$

with continuously differentiable functions  $a, b$  having bounded derivatives, can be obtained via discrete time approximation. Namely, if we define the sequence  $\{x_n^m\}$  by the rule:

$$x_0^m = x_0 \in \mathbb{R}, \quad x_{n+1}^m = x_n^m + \frac{1}{m}a(x_n^m) + \frac{1}{\sqrt{m}}b(x_n^m)\xi_n \quad (2)$$

where  $\{\xi_n, n \geq 1\}$  is a sequence of independent standard Gaussian random variables, then the random functions

$$x_m(t) = m \left( \frac{k+1}{m} - t \right) x_k^m + m \left( t - \frac{k}{m} \right) x_{k+1}^m, \quad t \in \left[ \frac{k}{m}; \frac{k+1}{m} \right], \quad k = 0, \dots, m-1$$

weakly converge in  $C([0, 1])$  to the solution of (1).

In this paper we study similar to (2) difference approximation for coalescing stochastic flows. As is known [2], such flows are not generated by a

Gaussian white noise in the space of vector fields. In order to understand how the flow with coalescence is arranged we can consider its difference approximation. As a perturbation we select a sequence of Gaussian stationary processes. In order to allow the coalescence of the trajectories of individual particles in the limit, the covariance functions of these processes are chosen to be less and less smooth at the origin. On the other hand, in order the limit flow to preserve the order, the step of approximation must be sufficiently small. The relationship between the step of approximation and smoothness of the covariance of the perturbing processes explains to some extent the structure of singular stochastic flows.

**1. SDE and stochastic flows on the real line.** The main object of the article is the Harris flow of Brownian motions on  $\mathbb{R}$ . Let  $\varphi$  be a continuous real positive definite function on  $\mathbb{R}$  such that  $\varphi(0) = 1$  and  $\varphi$  is Lipschitz outside any neighborhood of zero.

**Definition 1.** The Harris flow with  $\varphi$  being its local characteristic is a family  $\{x(u, \cdot); u \in \mathbb{R}\}$  of Brownian martingales with respect to the joint filtration such, that

1) for every  $u_1 \leq u_2$  and  $t \geq 0$

$$x(u_1, t) \leq x(u_2, t),$$

2) the joint characteristics are:

$$d\langle x(u_1, \cdot), x(u_2, \cdot) \rangle(t) = \varphi(x(u_1, t) - x(u_2, t))dt.$$

It is known that the Harris flow exists [3]. If the function  $\varphi$  is smooth enough, the Harris flow can be obtained as a flow of solutions to SDE. Namely, for a sequence of standard Wiener processes  $\{w_k; k \geq 1\}$  consider the following SDE

$$dx(u, t) = \sum_{k=1}^{\infty} a_k(x(u, t))dw_k(t), \tag{3}$$

where  $a = (a_k)_{k \geq 1}$  is a Lipschitz mapping from  $\mathbb{R}$  to  $l_2$  such that

$$\sum_{k=1}^{\infty} a_k^2 \equiv 1,$$

and

$$\sum_{k=1}^{\infty} a_k(u) a_k(v) = \varphi(u - v).$$

Then the flow corresponding to (3) is the Harris flow with the local characteristic  $\varphi$ , and furthermore it is a flow of homeomorphisms. Note, that the Harris flow could be coalescent [3] and, in this case may not be generated by SDE. By this reason it is interesting to consider discrete approximations for the flow built in a similar way as approximations to SDE. Consider a sequence of independent stationary Gaussian processes  $\{\xi_n(u); u \in \mathbb{R}, n \geq 1\}$  with zero mean and covariation function  $\Gamma$ . Suppose, that  $\Gamma$  is continuous. Define a sequence of random mappings  $\{x_n; n \geq 0\}$  by the rule

$$x_0(u) = u, \quad x_{n+1}(u) = x_n(u) + \xi_{n+1}(x_n(u)), \quad u \in \mathbb{R}. \quad (4)$$

Note, that the continuity of  $\Gamma$  implies that the processes  $\{\xi_n; n \geq 1\}$  have measurable modifications. This allows to substitute  $x_n$  into  $\xi_{n+1}$ . The independence of  $\{\xi_n; n \geq 1\}$  guarantees that  $\xi_{n+1}(x_n(u))$  does not depend on the choice of these modifications. We will need the following description of one and two-point motions of  $\{x_n; n \geq 0\}$ .

**Lemma 1.** *The sequences  $\{x_n(u); n \geq 0\}$  and  $\{x_n(u_2) - x_n(u_1); n \geq 0\}$  have the same distributions as the sequences  $\{y_n(u); n \geq 0\}$ ,  $\{z_n(u); n \geq 0\}$ , which are defined by the following rules:*

$$y_0 = u, \quad y_{n+1} = y_n + \eta_n,$$

$$z_0 = u_2 - u_1, \quad z_{n+1} = z_n + \sqrt{2\Gamma(0) - 2\Gamma(z_n)} \eta_n,$$

where  $\{\eta_n; n \geq 1\}$  is a sequence of independent standard normal variables.

The proof of the lemma can be obtained easily by calculating conditional distributions of  $x_{n+1}$  under given  $x_0, \dots, x_n$ , and is omitted.

It follows from Lemma 1, that the sequence of random mappings  $\{x_n; n \geq 0\}$  is similar to the Harris flow. All its one-point motions are Gaussian symmetric random walks. But the mappings  $x_n$  for  $n \geq 1$  are not monotone. In the next section we will prove that any  $m$ -point motion of  $\{x_n; n \geq 0\}$  approximates the  $m$ -point motion of the Harris flow.

**2.  $m$ -point motions.** In this section we will consider the limit behavior of  $x_n$  under a suitable normalization. Let us define the random functions

$$\tilde{x}_n(u, t) = n \left( \frac{k+1}{n} - t \right) x_k(u) + n \left( t - \frac{k}{n} \right) x_{k+1}(u),$$

$$u \in \mathbb{R}, t \in \left[ \frac{k}{n}; \frac{k+1}{n} \right], k = 0, \dots, n-1.$$

Our first result is related to the  $n$ -point motions of  $\tilde{x}_n$ .

**Theorem 1.** *Let  $\Gamma$  be continuous positive definite function on  $\mathbb{R}$  such that  $\Gamma(0) = 1$  and  $\Gamma$  has two continuous bounded derivatives. Suppose that  $\tilde{x}_n$  is built upon a sequence  $\{\xi_k; k \geq 1\}$  with covariance  $\frac{1}{\sqrt{n}}\Gamma$ .*

*Then for every  $u_1, \dots, u_l \in \mathbb{R}$  the random processes  $\{\tilde{x}_n(u_j, \cdot), j = 1, \dots, l\}$  weakly converge in  $C([0; 1], \mathbb{R}^l)$  to the  $l$ -point motion of the Harris flow with the local characteristic  $\Gamma$ .*

*Proof.* It follows from Lemma 1 and the invariance principle, that for every  $j = 1, \dots, l$   $\tilde{x}_n(u_j, \cdot)$  weakly converges in  $C([0; 1])$  to the Brownian motion which starts from  $u_j$ . Then, it remains to prove that any limit point of  $\{\tilde{x}_n(u_j, \cdot), j = 1, \dots, l\}$  coincides with the  $l$ -point motion of the Harris flow. Without loss of generality suppose that the whole sequence  $\{\tilde{x}_n(u_j, \cdot), j = 1, \dots, l\}$  weakly converges. For a function  $f \in C^3(\mathbb{R})$  with bounded derivatives, consider the random processes

$$y_n(t) = \tilde{x}_n(u_{j+1}, t) - \tilde{x}_n(u_j, t),$$

$$z_n(t) = f(y_n(t)) - f(u_{j+1} - u_j) - \int_0^t (1 - \Gamma(y_n(s))) f''(s) ds.$$

Following the known procedure (see for example [4]), it is easy to verify, that  $\{z_n; n \geq 1\}$  weakly converges to a certain martingale. Consequently the weak limit of  $y_n$  satisfies the martingale problem for the operator

$$Af(x) = (1 - \Gamma(x)) \frac{d^2}{dx^2} f(x).$$

Since the martingale problem now has a unique solution [4], then the weak limit of  $y_n$  is the solution to the following Cauchy problem

$$\begin{cases} dy(t) = \sqrt{2 - 2\Gamma(y(t))} dw(t), \\ y(0) = u_{j+1} - u_j. \end{cases}$$

The solution to this SDE has the strong Markov property. Consequently  $y$  is nonnegative for  $u_{j+1} - u_j > 0$ . Hence, the weak limit of  $\{\tilde{x}_n(u_j, \cdot); j = 1, \dots, l\}$  preserves the order. It remains to check the form of the joint characteristic, which can be done in a standard way. The theorem is proved.  $\square$

The previous result is based on the uniqueness of a solution to SDE related to a stochastic flow. Now we consider the convergence of difference approximations to the  $n$ -point motions of the Arratia flow. Let us recall that Arratia's flow [5] is the Harris flow with the local characteristic  $\Gamma = \mathbb{I}_{\{0\}}$ . In this flow any two trajectories coalesce into a single one in finite time.

**Theorem 2.** *Suppose, that for every  $m \geq 1$   $\tilde{x}_m$  is built upon a sequence  $\{\xi_n^m; n \geq 1\}$  where independent identically distributed processes  $\xi_n^m$  have the covariance function  $\Gamma_m$  which satisfies the Lipschitz condition. Define for  $m \geq 1$*

$$C_m = \sup_{\mathbb{R}} \frac{2 - 2\Gamma_m(x)}{x^2}.$$

*If*

$$1) \lim_{m \rightarrow \infty} \frac{C_m e^{C_m}}{m} = 0,$$

$$2) \text{ for every } \delta > 0 \sup_{\mathbb{R} \setminus [-\delta; \delta]} |\Gamma_m(x)| \rightarrow 0, m \rightarrow \infty,$$

*then the random processes  $\{\tilde{x}_m(u_1, \cdot), \dots, \tilde{x}_m(u_l, \cdot); m \geq 1\}$  weakly converge to the  $l$ -point motion of Arratia's flow starting from  $u_1, \dots, u_l$ .*

*Proof.* As in the proof of Theorem 1 we have the weak compactness of  $\{(\tilde{x}_m(u_1, \cdot), \dots, \tilde{x}_m(u_l, \cdot); m \geq 1)\}$  in  $C([0; 1], \mathbb{R}^l)$  and the weak convergence of  $x_m(u_i, \cdot)$  to a Wiener process. Consequently, for any limit point of  $\{(\tilde{x}_m(u_1, \cdot), \dots, \tilde{x}_m(u_l, \cdot); m \geq 1)\}$  it is enough to check the mutual characteristics and the order preserving property. For  $u_i < u_{i+1}$  the difference process  $y_m(t) = \tilde{x}_m(u_{i+1}, t) - \tilde{x}_m(u_i, t)$  are equidistributed with the difference approximation  $v_m$  to the solution of the SDE

$$\begin{cases} d\tilde{y}_m(t) = \sqrt{2 - 2\Gamma_m(\tilde{y}_m(t))} dw(t), \\ \tilde{y}_m(0) = u_{i+1} - u_i. \end{cases}$$

It is known [1], that

$$E \sup_{[0; 1]} (v_m(t) - \tilde{y}_m(t))^2 \leq C \frac{C_m e^{C_m}}{m}.$$

Note, that  $\tilde{y}_m$  is nonnegative. Consequently, for every  $r > 0$

$$P\{\inf_{[0;1]} y_m < -r\} = P\{\inf_{[0;1]} v_m < -r\} \rightarrow 0, m \rightarrow \infty.$$

Hence the weak limit of any subsequence of  $\{y_m; m \geq 1\}$  is nonnegative. The completion of the proof can be done exactly as in the previous theorem using martingale approximation and the fact that any nonnegative martingale remains at zero after hitting zero. The theorem is proved.  $\square$

### 3. Convergence of random maps

In this section we will consider convergence of  $\{\tilde{x}_n; n \geq 1\}$  as random maps to corresponding maps from a stochastic flow. Let us begin with the case of smooth  $\Gamma$ . Define the sequence

$$x_{n+1}^m(u) = x_n^m(u) + \frac{1}{\sqrt{m}} \xi_{n+1}(x_n^m(u)), \quad (5)$$

where  $\{\xi_n; n \geq 1\}$  is a sequence of independent stationary centered Gaussian processes with covariance function  $\Gamma$  satisfying the inequality

$$\forall u \in \mathbb{R} : 1 - \Gamma(u) \leq Cu^2$$

with some constant  $C$ . Define the Harris flow  $x$  corresponding to  $\Gamma$ . Note, that now  $x$  has a modification  $x(u, t), u \in \mathbb{R}, t \in [0; 1]$  continuous with respect to both variables. Really, using the martingale inequality one can get that

$$\begin{aligned} E \sup_{s \in [0; t]} (x(u, s) - x(v, s))^2 &\leq \\ &\leq 2(u - v)^2 + 2E \int_0^t (2 - 2\Gamma(x(u, s) - x(v, s))) ds \leq \\ &\leq 2(u - v)^2 + 4C \int_0^t \sup_{r \in [0; s]} (x(u, r) - x(v, r))^2 ds. \end{aligned}$$

Consequently, for some  $\tilde{c}$

$$E \sup_{t \in [0; 1]} (x(u, t) - x(v, t))^2 \leq \tilde{c}(u - v)^2.$$

This inequality together with the Kolmogorov condition gives us the desired property.

The next statement asserts the convergence of our approximations to a stochastic flow in the case of smooth  $\Gamma$ .

**Theorem 3.** *The random functions  $\{\tilde{x}_m = x_m^m; m \geq 1\}$  converge in distribution in the space  $C([a; b])$  to the random function  $x$  for arbitrary interval  $[a; b]$ .*

*Proof.* The convergence of finite-dimensional distributions was proved in Theorem 1. It remains to check the weak compactness of  $\{\tilde{x}_m; m \geq 1\}$ . For arbitrary  $u, v \in \mathbb{R}$  we have

$$\begin{aligned} E(x_{n+1}^m(u) - x_{n+1}^m(v))^2 &= E(x_n^m(u) - x_n^m(v))^2 + \frac{1}{m} E(2 - 2\Gamma(x_n^m(u) - x_n^m(v))) \leq \\ &\leq E(x_n^m(u) - x_n^m(v))^2 + 2Cm^{-1} E(x_n^m(u) - x_n^m(v))^2. \end{aligned}$$

Consequently,

$$E(\tilde{x}_m(u) - \tilde{x}_m(v))^2 \leq (u - v)^2 (1 + \frac{2C}{m})^m \leq e^{2C} (u - v)^2.$$

The obtained estimation gives the desired weak compactness. The theorem is proved.  $\square$

To obtain approximation of Arratia's flow we need some additional results about the convergence of smooth stochastic flows to Arratia's flow. Let us consider the following SDE with the space-time white noise (Wiener sheet)  $W$

$$\begin{aligned} dz(u, t) &= \int_{\mathbb{R}} \varphi(z(u, t) - p) W(dp, dt), \\ z(u, 0) &= u, \quad u \in \mathbb{R}, \end{aligned} \tag{6}$$

where  $\varphi \in C_0^\infty(\mathbb{R})$  and  $\int_{\mathbb{R}} \varphi^2(u) du = 1$  (see [6, 7] about equations of type (6)). All what we need here is a statement, that under our condition on  $\varphi$  the unique strong solution to (6) exists and is the Harris flow corresponding to the local characteristic

$$\Gamma(u) = \int_{\mathbb{R}} \varphi(-p) \varphi(u - p) dp.$$

It was proved in [8], that the  $n$ -point motions of solutions  $z_\varepsilon$  to (6) which corresponds to  $\varphi_\varepsilon$  with the property  $\text{supp } \varphi_\varepsilon \subset [-\varepsilon; \varepsilon]$  converge in distribution to the  $n$ -point motions of the Arratia flow when  $\varepsilon \rightarrow 0$ .

Consider discrete approximations of  $z$ . For every  $n \geq 1$  define

$$\begin{aligned} z_0^n(u) &= u, \\ z_{k+1}^n(u) &= z_k^n(u) + \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\mathbb{R}} \varphi(z_k^n(u) - p) W(dp, dt), \\ k &= 0, \dots, n-1. \end{aligned} \quad (7)$$

It can be easily checked that every  $z_k^n$  has a continuous modification. The next theorem gives a speed of convergence of  $z_n^n$  to  $z(\cdot, 1)$  in the space  $C([0; 1])$ . Define

$$L^2 = \int_{\mathbb{R}} \varphi'(p)^2 dp.$$

**Theorem 4.** *There exist such positive constants  $C', C'', C'''$ , that for every  $n \geq 1$*

$$E \|z_n^n - z(\cdot, 1)\| \leq \frac{C'}{\sqrt{n}} \exp\{(C'' L^2 + C''' L^4) e^{4L^2} + L^2\} (L^2 + 1). \quad (8)$$

where  $\|\cdot\|$  is the uniform norm in  $C([0; 1])$ .

*Proof.* Consider for  $k = 1, \dots, n$

$$\begin{aligned} E \left( z_k^n(0) - z \left( 0, \frac{k}{n} \right) \right)^2 &= E \left( z_{k-1}^n(0) - z \left( 0, \frac{k-1}{n} \right) \right)^2 + \\ &+ E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} (\varphi(z_{k-1}^n(0) - p) - \varphi(z(0, s) - p))^2 dp ds \leq \\ &\leq E \left( z_{k-1}^n(0) - z \left( 0, \frac{k-1}{n} \right) \right)^2 + L^2 E \int_{\frac{k-1}{n}}^{\frac{k}{n}} (z_{k-1}^n(0) - z(0, s))^2 ds = \\ &= E \left( z_{k-1}^n(0) - z \left( 0, \frac{k-1}{n} \right) \right)^2 \left( 1 + L^2 \frac{1}{n} \right) + \\ &+ L^2 E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( z(0) - z \left( 0, \frac{k-1}{n} \right) \right)^2 ds = \\ &= E \left( z_{k-1}^n(0) - z \left( 0, \frac{k-1}{n} \right) \right)^2 \left( 1 + \frac{L^2}{n} \right) + \frac{L^2}{2n^2}. \end{aligned}$$



Consequently,

$$E(z_n^n(0) - z(0, 1))^2 \leq \frac{L^2}{n^2} e^{L^2}.$$

Note, that under our conditions on  $\varphi$ , random functions  $\{z_k^n\}$  and  $z$  have continuous derivatives with respect to the spatial variable.

Let us denote by  $y_k^n$  and  $y$  these derivatives. Then for  $k = 1, \dots, n$

$$y_k^n(u) = y_{k-1}^n(u) \left( 1 + \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} \varphi'(z_{k-1}^n(u) - p) W(dp, dt) \right),$$

and

$$dy(u, t) = y(u, t) \int_{\mathbb{R}} \varphi'(z(u, t) - p) W(dp, dt).$$

Hence,

$$\begin{aligned} y_k^n(u) - y\left(u, \frac{k}{n}\right) &= y_{k-1}^n(u) - y\left(u, \frac{k}{n}\right) + \\ &+ \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} [y_{k-1}^n(u) \varphi'(z_{k-1}^n(u) - p) - y(u, s) \varphi'(z(u, s) - p)] W(dp, dt) = \\ &= y_{k-1}^n(u) - y\left(u, \frac{k-1}{n}\right) + \\ &+ \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} \left[ \left( y_{k-1}^n(u) - y\left(u, \frac{k-1}{n}\right) \right) \varphi'(z_{k-1}^n(u) - p) + \right. \\ &+ y\left(u, \frac{k-1}{n}\right) (\varphi'(z_{k-1}^n(u) - p) - \varphi'\left(z\left(u, \frac{k-1}{n}\right) - p\right) + \\ &+ y\left(u, \frac{k-1}{n}\right) \left( \varphi'\left(z\left(u, \frac{k-1}{n}\right) - p\right) - \varphi'(z(u, s) - p) \right) + \\ &\left. + \varphi'(z(u, s) - p) \left( y\left(u, \frac{k-1}{n}\right) - y(u, s) \right) \right] W(dp, ds). \end{aligned}$$

Then

$$\begin{aligned} E\left(y_k^n(u) - y\left(u, \frac{k}{n}\right)\right)^2 &= E\left(y_{k-1}^n(u) - y\left(u, \frac{k-1}{n}\right)\right)^2 + \\ &+ 4E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} \left( y_{k-1}^n(u) - y\left(u, \frac{k-1}{n}\right) \right)^2 \varphi'(z_{k-1}^n(u) - p)^2 dp ds + \end{aligned}$$

$$\begin{aligned}
& +4E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} y \left( u, \frac{k-1}{n} \right)^2 (\varphi'(z_{k-1}^n(u)-p) - \varphi' \left( z \left( u, \frac{k-1}{n} \right) - p \right))^2 dp ds + \\
& +4E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} y \left( u, \frac{k-1}{n} \right)^2 \left( \varphi' \left( z \left( u, \frac{k-1}{n} \right) - p \right) - \varphi'(z(u, s) - p) \right)^2 dp ds + \\
& +4E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\mathbb{R}} \varphi'(z(u, s) - p)^2 \left( y \left( u, \frac{k-1}{n} \right) - y(u, s) \right)^2 dp ds \leq \\
& \leq E \left( y_{k-1}^n(u) - y \left( u, \frac{k-1}{n} \right) \right)^2 \cdot (1 + \frac{4}{n} L^2) + \\
& + \frac{4}{n} E y \left( u, \frac{k-1}{n} \right)^2 \cdot L^2 \left( z_{k-1}^n(u) - z \left( u, \frac{k-1}{n} \right) \right)^2 + \\
& + 4E y \left( u, \frac{k-1}{n} \right)^2 L^2 \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( z \left( u, \frac{k-1}{n} \right) - z(u, s) \right)^2 ds + \\
& + 4L^2 E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( y \left( u, \frac{k-1}{n} \right) - y(u, s) \right)^2 ds.
\end{aligned}$$

Note, that the processes  $z(u, t), t \in [0; 1]$  and

$$\eta(t) = \int_0^t \int_{\mathbb{R}} \varphi'(z(u, s) - p) W(dp, ds), \quad t \in [0; 1]$$

are continuous martingales with the characteristics

$$\langle z(u, \cdot) \rangle(t) = t, \quad \langle \eta \rangle(t) = L^2 t.$$

Consequently,  $z(u, \cdot)$  and  $\eta$  are Wiener processes. It follows from this that

$$y(u, t) = \exp\{\eta(t) - \frac{t}{2} L^2\}.$$

Hence

$$\begin{aligned}
& E y \left( u, \frac{k-1}{n} \right)^2 \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( z \left( u, \frac{k-1}{n} \right) - z(u, s) \right)^2 ds = \\
& = E y \left( u, \frac{k-1}{n} \right)^2 E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( z \left( u, \frac{k-1}{n} \right) - z(u, s) \right)^2 ds \leq \\
& \leq \frac{1}{2n^2} e^{L^2},
\end{aligned}$$

$$\begin{aligned}
& E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( y \left( u, \frac{k-1}{n} \right) - y(u, s) \right)^2 ds = \\
& = E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left( \int_{\frac{k-1}{n}}^s \int_{\mathbb{R}} y(u, r) \varphi'(z(u, r) - p) W(dp, dr) \right)^2 ds = \\
& = L^2 E \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^s y(u, r)^2 dr ds \leq \\
& \leq \frac{1}{2n^2} L^2 \cdot e^{L^2}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& E y \left( u, \frac{k-1}{n} \right)^2 \left( z_{k-1}^n(u) - z \left( u, \frac{k-1}{n} \right) \right)^2 \leq \\
& \leq \sqrt{E y \left( u, \frac{k-1}{n} \right)^4} \sqrt{E \left( z_{k-1}^n(u) - z \left( u, \frac{k-1}{n} \right) \right)^4} \leq \\
& \leq e^{3L^2} C_2 \frac{L^2}{n} e^{L^2}.
\end{aligned}$$

In the last inequality the martingale property of  $x^n$  and  $x$  was used. Finally one can get

$$\begin{aligned}
& E \left( y_k^n(u) - y \left( u, \frac{k}{n} \right) \right)^2 \leq \\
& \leq E \left( y_{k-1}^n(u) - y \left( u, \frac{k}{n} \right) \right)^2 \left( 1 + \frac{4L^2}{n} + \frac{4}{n} L^4 C_2 e^{4L^2} \right) + \frac{c_3}{n^2} (L^2 + 1) e^{L^2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& E(y_n^n(u) - y(u, 1))^2 \leq \\
& \leq \frac{c_4}{n} \exp\{(c_5 L^2 + c_6 L^4) e^{4L^2} + L^2\} (L^2 + 1).
\end{aligned}$$

To obtain an estimation for the uniform norm  $\|z_n^n - z(\cdot, 1)\|$  we proceed as follows

$$\begin{aligned}
E\|z_n^n - z(\cdot, 1)\| & \leq E|z_n^n(0) - z(0, 1)| + E \int_0^1 |y_n^n(u) - y(u, 1)| du \leq \\
& \leq \frac{c_7}{\sqrt{n}} \exp\{(c_8 L^2 + c_9 L^4) e^{4L^2} + L^2\} (L^2 + 1).
\end{aligned}$$

The theorem is proved. □

Obtained estimation can be used to get the convergence of the difference approximation to Arratia's flow. We will establish this convergence using the Lévy–Prokhorov distance. Let us recall its definition.

**Definition 2.** [9]. For two nondecreasing càdlàg functions  $f, g$  on  $[0; 1]$  the Lévy–Prokhorov distance is

$$\begin{aligned}\rho(f, g) &= \inf\{\varepsilon > 0 : \forall u \in [0; 1] : \\ &\quad f(u - \varepsilon) - \varepsilon \leq g(u) \leq f(u + \varepsilon) + \varepsilon \\ &\quad g(u - \varepsilon) - \varepsilon \leq f(u) \leq g(u + \varepsilon) + \varepsilon\}.\end{aligned}$$

It is well-known [9] that the convergence in this distance is equivalent to the convergence at every point of continuity of the limit function. Also note that

$$\rho(f, g) \geq d(f, g),$$

where  $d(f, g)$  is the Skorokhod distance between  $f$  and  $g$  [9].

Take a function  $\psi \in C_0^\infty$  with  $\text{supp } \psi \subset [-1; 1]$  such, that

$$\int_{\mathbb{R}} \psi^2(u) du = 1.$$

For arbitrary  $\varepsilon > 0$  define

$$\begin{aligned}\psi_\varepsilon(u) &= \frac{1}{\varepsilon^{1/2}} \psi\left(\frac{u}{\varepsilon}\right), \\ \Gamma_\varepsilon(u) &= \frac{1}{\varepsilon} \int_{\mathbb{R}} \psi_\varepsilon(p) \psi_\varepsilon(u + p) dp.\end{aligned}$$

The parameter  $\varepsilon$  here is associated with the smoothness of  $\Gamma_\varepsilon$ . In order to approximate the Arratia flow we have to take  $\varepsilon \rightarrow 0$ . For independent Gaussian processes  $\{\xi_n; n \geq 1\}$  with the covariance  $\{\Gamma_{\varepsilon_n}\}$  let us construct the sequences

$$x_{k+1}^n(u) = x_k^n(u) + \frac{1}{\sqrt{n}} \xi_n(x_k^n(u)).$$

The next theorem shows that  $x_n^n$  can be used to approximate the Arratia flow.

**Theorem 5.** *Suppose that  $\varepsilon_n \rightarrow 0, n \rightarrow \infty$ ,*

$$\frac{1}{\varepsilon_n^2} = o(\ln n), \quad n \rightarrow \infty.$$

*Then the random functions  $x_n^n$  converge weakly in  $D([0; 1])$  to the value of the Arratia flow  $x(\cdot, 1)$ .*

*Proof.* Consider the sequence of SDE

$$dz_{\varepsilon_n}(u, t) = \int_{\mathbb{R}} \psi_{\varepsilon}(z_{\varepsilon_n}(u, t) - p) W(dp, dt).$$

As it was mentioned in the beginning of this section, for every  $u_1, \dots, u_m \in [0; 1]$  ( $z_{\varepsilon_n}(u_1, 1), \dots, z_{\varepsilon_n}(u_m, 1)$ ) weakly converge to  $(x(u_1, 1), \dots, x(u_m, 1))$ . Hence [10],  $z_{\varepsilon_n}(\cdot, 1)$  weakly converge to  $x(\cdot, 1)$  in the Lévy–Prokhorov distance. For every  $n \geq 1$  the sequence  $x_1^n, \dots, x_n^n$  is equidistributed with the discrete approximations to  $z_{\varepsilon_n}$  from Theorem 4. Consequently,  $x_n^n$  is equidistributed with  $\tilde{x}_n$  such that

$$E\|\tilde{x}_n - z_{\varepsilon_n}(\cdot, 1)\| \leq \frac{C'}{\sqrt{n}} \exp\{(C''L_{\varepsilon_n}^2 + C'''L_{\varepsilon_n}^4)e^{4L_{\varepsilon_n}^2} + L_{\varepsilon_n}^2\}(L_{\varepsilon_n}^2 + 1),$$

where  $L_{\varepsilon_n}^2 = \frac{1}{\varepsilon_n} \int_{\mathbb{R}} \psi'(p)^2 dp$ . Hence

$$E\|\tilde{x}_n - z_{\varepsilon_n}(\cdot, 1)\| \rightarrow 0, n \rightarrow \infty.$$

Since for continuous functions  $f, g$  the Skorokhod distance

$$d(f, g) \leq \|f - g\|,$$

then  $x_n^n$  weakly converges to  $x(\cdot, 1)$  in  $D([0; 1])$ . The theorem is proved.  $\square$

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